AN ALGEBRAIC APPROACH TO KNOWLEDGE BASES INFORMATIONAL EQUIVALENCE

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Abstract:

In this paper we study the notion of knowledge from the positions of universal algebra and algebraic logic. We consider first order knowledge which is based on first order logic. We define categories of knowledge and knowledge bases. These notions are defined for the fixed subject of knowledge. The key notion of informational equivalence of two knowledge bases is introduced. We use the idea of equivalence of categories in this definition. We prove that for finite models there is a clear way to determine whether the knowledge bases are informationally equivalent.

Introduction

This work stands at intersection of two areas: universal algebra and category theory on the one hand and a field we call knowledge science on the other. We view the latter as a science dealing with languages of knowledge representation. It is strongly related to universal algebra and can be considered as an area of mathematics having motivation in computer science.

Knowledge theory and knowledge bases provide an important example of the field where application of universal algebra and algebraic logic is very natural, and

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their interacting with quite practical problems arising in computer science is very productive. Another examples of such interaction are given by relational database theory, constraint satisfaction problem ([BJ],[JCP]), theory of complexity, and by others.

One can speak about knowledge and a system of knowledge. As a rule, a domain of knowledge or of a system of knowledge is fixed. In our approach only knowledge that allows a formalization in some logic is considered. The logic may be different. It is often oriented towards the corresponding field of knowledge cf. [G],[L],[S].

In this paper we focus on the special situation of elementary knowledge.

Elementary knowledge is considered to be a first order knowledge, i.e., the knowledge that can be represented by the means of the First Order Logic (FOL). The corresponding applied field (field of knowledge) is grounded on some variety of algebras Θ, which is arbitrary but fixed. This variety Θ is considered as a knowledge type. Its counterpart in database theory is the notion of datatype Θ.

We also fix a set of symbols of relations Φ. The subject of knowledge is a triple (G, Φ, f), where G is an algebra in Θ and f is a interpretation of the set Φ in G. It is a model in the ordinary mathematical sense. As a rule, we use shorthand and write f instead of (G, Φ, f). For the given Φ we denote the corresponding applied field by $\Phi\Theta$.

FOL is also oriented on the variety Θ.

We assume that every knowledge under consideration is represented by three components:

1) *The description of knowledge.* It is a syntactical part of knowledge, written out in the language of the given logic. The description reflects, what do we want to know.

2) *The subject of knowledge* which is an object in the given applied field, i.e., an object for which we determine knowledge.

3) *The content of knowledge* (its semantics).

The first two components are relatively independent, while the third one is uniquely determined by the previous two. In the theory under consideration, this third component has a geometrical nature. In some sense it is an algebraic set in an affine space. If T is a description of knowledge and (G, Φ, f) is a subject, then T^f

denotes the content of knowledge. We would like to equip the content with its own structure, algebraic or geometric, and to consider some aspects of such structure.

We want to underline that there are three aspects in our approach to knowledge representation: logical (for knowledge description), algebraic (for the subject of knowledge) and geometric (in the content of knowledge). This geometry is of algebraic nature. However, the involved algebra inherits some geometric intuition.

Let us emphasize that logic (syntax) and geometry (semantics) often interlace: its own geometry is possible in logic, while logic is possible in geometry. In general, we can eliminate geometry and reduce everything to logic. But this leads to essential loss, namely we loose geometrical intuition which supplements logic.

We consider categories of elementary knowledge. The language of categories in knowledge theory is a good way to organize and systematize primary elementary knowledge. Morphisms in a knowledge category give links between knowledge. In particular, one can speak of isomorphic knowledge. The categorical approach also allows us to use ideas of monada and comonada [ML]. It turns out that this provides some general views on enrichment and computation of knowledge. Enrichment of a structure can be associated with a suitable monada over a category, while the corresponding computation is organized by comonada. A knowledge base is related to a category of knowledge.

This paper is in a sense a continuation of [PTP]; we repeat some material to make the paper self-contained. However, there are certain changes in the approach to the basic notions in comparison to [PTP]. The main one is that the definition of knowledge bases (KBs) equivalence uses the idea of categories equivalence. To every KB it corresponds a database (DB)[PTP]. According to the principal result of the paper in the situation of finite models KBs are equivalent if and only if the corresponding databases are equivalent. This result is contained in the main Theorem 4 of the paper.

The paper is organized as follows. We include the material from [PTP] which is necessary for the understanding of the further sections: the first four sections follow [PTP] and provide a background to what follows. For the details see [Pl1], [Pl2],[Pl3].

1. ALGEBRA AND LOGIC

1.1 Multi-sorted algebra. Keeping in mind applications, throughout the paper the term algebra means multi-sorted, i.e., not necessarily one-sorted, algebra. We fix a set of sorts Γ. In the considered varieties Θ this set is finite, but it need not to be finite in general. We meet infinite Γ in the next section.

For every algebra $G \in \Theta$ we write

$$G = (G_i, i \in \Gamma).$$

The set of operations Ω is called the *signature of algebras* in Θ. Every symbol $\omega \in \Omega$ has a type $\tau = \tau(\omega) = (i_1, \dots, i_n; j), i, j \in \Gamma$. An operation of type τ is a mapping

$$G_{i_1} \times \ldots \times G_{i_n} \to G_j.$$

All operations of the signature Ω satisfy some set of identities. These identities define the variety Θ of Γ-sorted Ω-algebras. Let us consider homomorphisms and free algebras in Θ . A homomorphism of algebras in Θ has the form

$$\mu = (\mu_i, i \in \Gamma) \colon G = (G_i, i \in \Gamma) \to G' = (G'_i, i \in \Gamma).$$

Here $\mu_i \colon G_i \to G'_i$ are mappings of sets, coordinated with operations in Ω. A congruence $Ker\mu = (\mathrm{Ker}\mu_i, i \in \Gamma)$ is the kernel of a homomorphism μ.

We consider multi-sorted sets $X = (X_i, i \in \Gamma)$ and the corresponding free in Θ algebras

$$W = W(X) = (W_i, i \in \Gamma).$$

A set X and a free algebra W can be presented as free union of all X_i and all W_i, respectively.

Every (multi-sorted) mapping $\mu : X \to G$ is extended up to a homomorphism $\mu : W \to G$. Denote the set of all such μ by $\mathrm{Hom}(W, G)$. If all X_i are finite, we treat this set as an affine space. Homomorphisms $\mu \colon W \to G$ are points of this space.

For the given $G = (G_i, i \in \Gamma)$ and $X = (X_i, i \in \Gamma)$ we can consider the set

$$G^X = (G_i^{X_i}, i \in \Gamma)$$

It is the set of mappings

$$\mu = (\mu_i, i \in \Gamma) \colon X \to G.$$

There is a natural bijection $\mathrm{Hom}(W, G) \to G^X$. More information about multi-sorted algebras can be found in [Pl1].

Now let us turn to the models. Fix some set of symbols of relations Φ. Every $\varphi \in \Phi$ has its type $\tau = \tau(\varphi) = (i_1, \dots, i_n)$. A relation, corresponding to φ, is a subset in the Cartesian product $G_{i_1} \times \ldots \times G_{i_n}$. Denote by $\Phi\Theta$ the class of models (G, Φ, f), where $G \in \Theta$, and f is a interpretation of the set Φ in G. As for homomorphisms of models, they are homomorphisms of the corresponding algebras which are coordinated with relations.

1.2 Logic. We consider logic in the given variety Θ. For every finite X, there is a logical signature

$$L = L_X = \{\vee, \wedge, \neg, \exists x, \ \ x \in X\},$$

where X is $\bigcup\limits_{i \in \Gamma} X_i$ for a finite Γ. We consider the set (more precisely, the L-algebra) of formulas $L\Phi W$ over the free algebra $W = W(X)$. This algebra is an L-algebra of formulas of FOL over the given Θ, Φ, and X.

First we define the atomic formulas. They are equalities of the form $w \equiv w'$, with $w, w' \in W$ of the same sort and the formulas $\varphi(w_1, \dots, w_n)$, where $w_i \in W$, and all w_i are positioned according to the type $\tau = \tau(\varphi)$ of the relations φ and to the sorts. The set of all atomic formulas we denote by $M = M_X$. Define $L\Phi W$ to be the absolutely free L_X-algebra over M_X.

Let us consider another example of an L_X-algebra.

Given $W = W(X)$ and $G \in \Theta$, denote by $\mathrm{Bool}(W, G)$ the Boolean algebra $\mathrm{Sub}(\mathrm{Hom}(W, G))$ of all subsets in $\mathrm{Hom}(W, G)$. Define the action of quantifiers in $\mathrm{Bool}(W, G)$. Let A be a subset in $\mathrm{Hom}(W, G)$ and $x \in X_i$ be a variable of the sort i. Then $\mu \colon W \to G$ belongs to the set $\exists x A$ if there exists $\nu \colon W \to G$ in A such that $\mu(y) = \nu(y)$ for every $y \in X$ of the sort $j, j \neq i$, and for every $y \in X_i$, $y \neq x$. Thus we get an L-algebra $\mathrm{Bool}(W, G)$.

Now let us define a mapping

$$\mathrm{Val}^X \colon M_X \to \mathrm{Bool}(W, G)$$

where f is a model (the subject of knowledge), which realizes the set Φ in the given G. If $w \equiv w'$ is an equality of the sort i, then we set:

$$\mu : W \to G \in \mathrm{Val}_f^X(w \equiv w') = \mathrm{Val}^X(w \equiv w')$$

if $\mu_i(w) = \mu_i(w')$ in G. Here the point μ is a solution of the equation $w \equiv w'$. If the formula is of the form $\varphi(w_1, \dots, w_n)$, then

$$\mu \in \mathrm{Val}_f^X(\varphi(w_1, \dots, w_n))$$

if $\varphi(\mu(w_1), \dots, \mu(w_n))$ is valid in the model (G, Φ, f). Here $\mu(w_j) = \mu_{i_j}(w_j)$, i_j is the sort of w_j. The mapping Val_f^X is uniquely extended up to the L-homomorphism

$$\mathrm{Val}_f^X : L\Phi W \to \mathrm{Bool}(W, G).$$

Thus, for every formula $u \in L\Phi W$ we defined its value $\mathrm{Val}_f(u)$ in the model (G, Φ, f), which is an element in $\mathrm{Bool}(W, G)$.

Every formula $u \in L\Phi W$ can be viewed as an equation in the given model. Then a point $\mu : W \to G$ is the solution of the "equation" u if $\mu \in \mathrm{Val}_f(u)$.

1.3 Geometrical Aspect.

In the L-algebra of formulas $L\Phi W$, $W = W(X)$, we consider its various subsets T. On the other hand, we consider subsets A in the affine space $\mathrm{Hom}(W, G)$, i.e., elements of the L-algebra $\mathrm{Bool}(W, G)$. For each model (G, Φ, f) and for these T and A we establish the following *Galois correspondence* between sets of formulas in L-algebra of formulas $L\Phi W$ and sets of points in the space $\mathrm{Hom}(W, G)$:

$$T^f = A = \bigcap_{u \in T} \mathrm{Val}_f(u),$$
$$A^f = T = \{u | A \subset \mathrm{Val}_f(u)\}.$$

Here $A = T^f$ is a locus of all points satisfying the formulas from T. We regard T also as a system of "equations", where each "equation" is represented by a formula u from T. Every set A of such kind is said to be an *algebraic set* (or closed set, or algebraic variety), determined for the given model. We define *knowledge* as

$$(X, T, A, (G, \Phi, f)).$$

Here T is a *description of knowledge* and (G, Φ, f) is a *subject of knowledge.* $A =$ T^f is a *content of knowledge* represented as an algebraic variety. X is a *place of*

knowledge (the place, where the knowledge is situated). A set A can be regarded also as a relation between elements of G derived from equalities and relations of the basic set Φ. The relation $A = T^f$ belongs to the multi-sorted set

$$G^X = \{G_i^{X_i}, \ \ i \in \Gamma\}.$$

A set T of the form $T = A^f$ for some A is called an *f-closed set*. For an arbitrary T we have its closure $T^{ff} = (T^f)^f$ and for every $A \subset \mathrm{Hom}(W, G)$ we have the closure $A^{ff} = (A^f)^f$.

It is easy to understand that the following rule takes place:

A formula v belongs to the set T^{ff} if and only if the formula

$$(\bigwedge_{u \in T} u) \to v$$

holds in the model (G, Φ, f).

If the set T is infinite then the corresponding formula is called *infinitary*.

We want to study knowledge with different, changing "places of knowledge" X. In this case one should consider different $W = W(X)$, different "spaces of knowledge" $\mathrm{Hom}(W(X), G)$, and different $L\Phi W(X)$.

Free in Θ algebras $W(X)$ with finite X are the objects of the category, denoted by Θ^0. Morphisms of this category $s \colon W(X) \to W(Y)$ are arbitrary homomorphisms of algebras. The category Θ^0 is a full subcategory in the category Θ.

We intend to build a new category related to the first order logic for the given Θ. This category will play for the FOL the role similar to that of the category of free algebras Θ^0 for the equational logic. With this end we turn from pure logic to algebraic logic. Such a transition will allow us to associate description of knowledge with its content in a more interesting way. The sets of the type $T = A^f$ also look more natural.

2. Algebraic logic

2.1 The main idea. Algebraic logic deals with algebraic structures, related to various logical structures which correspond to different logical calculi. For example,

are associated with non-classical propositional logic, Tarski cylindric algebras and Halmos polyadic algebras are associated with FOL.

Every logical calculus assumes that there are formulas of the calculus, axioms of logic and rules of inference. On this basis a syntactical equivalence of formulas compatible with their semantical equivalence is defined. The transition from pure logic to algebraic logic is grounded on treating logical formulas up to a certain equivalence. We call the corresponding classes the *compressed formulas*. This transition leads to various special algebraic structures, in particular to the structures mentioned above.

Every logical calculus is usually associated with some infinite set of variables. Denote such a set by X^0. In our situation it is a multi-sorted set $X^0 = (X_i^0, i \in \Gamma)$. Keeping in mind theory of knowledge and its geometrical aspect we will use a system of all finite subsets $X = (X_i,\ i \in \Gamma)$ of X^0 instead of this infinite universum. This gives rise to multi-sorted logic and multi-sorted algebraic logic. Every formula has a definite type (sort) X. Denote the new set of sorts by Γ^0. It is a set of all finite subsets of the initial set X^0.

2.2 Halmos Categories. Fix some variety of algebras Θ. This means that a finite set of sorts Γ, a signature $\Omega = \Omega(\Theta)$ related to Γ, and a system of identities $Id(\Theta)$ are given.

Define *Halmos categories* for the given Θ.

First, for the given Boolean algebra B we define its existential quantifiers [HMT]. Existential quantifiers are the mappings $\exists\colon B \to B$ with the conditions:

1) $\exists 0 = 0$,

2) $a < \exists a$,

3) $\exists (a \wedge \exists b) = \exists a \wedge \exists b$, $0, a, b \in B$.

The universal quantifier $\forall\colon B \to B$ is defined dually:

1) $\forall 1 = 1$,

2) $a > \forall a$,

3) $\forall (a \vee \forall b) = \forall a \vee \forall b$.

Let B be a Boolean algebra and X a set. We say that B is a *quantifier X-algebra* if a quantifier $\exists x\colon B \to B$ is defined for every $x \in X$ and for every two elements $x, y \in X$ the equality $\exists x \exists y = \exists y \exists x$ holds.

One may consider also *quantifier X-algebras B with equalities* over $W(X)$. In such algebras, to each pair of elements $w, w' \in W(X)$ of the same sort it corresponds an element $w \equiv w' \in B$ satisfying the conditions

1) $w \equiv w$ is the unit in B,

2) $(w_1 \equiv w'_1 \wedge \ldots \wedge w_n \equiv w'_n) < (w_1 \ldots w_n \omega \equiv w'_1 \ldots w'_n \omega)$ where ω is an operation in Ω and everything is compatible with the type of operation.

Now we will give the general definition of the Halmos category for the given Θ, which will be followed by examples.

Halmos category H for an arbitrary finite $X = (X_i, i \in \Gamma)$ fixes some quantifier X-algebra $H(X)$ with equalities over $W(X)$. $H(X)$ are the objects of H.

The morphisms in H correspond to morphisms in the category Θ^0. Every morphism s_* in H has the form

$$s_* = s \colon H(X) \to H(Y),$$

where $s \colon W(X) \to W(Y)$ is a morphism in Θ^0.

We assume that

1) The transitions $W(X) \to H(X)$ and $s \to s_*$ yield a (covariant) functor $\Theta^0 \to H$.

2) Every $s_* \colon H(X) \to H(Y)$ is a Boolean homomorphism.

3) The coordination with the quantifiers is as follows:

3.1) $s_1 \exists x a = s_2 \exists x a, \quad a \in H(X)$, if $s_1 y = s_2 y$ for every $y \in X,\ y \neq x$.

3.2) $s \exists x a = \exists (sx)(sa)$ if $sx = y \in Y$ and $y = sx$ is not in the support of sx', $x' \in X,\ x' \neq x$.

4) The following conditions describe coordination with equalities

4.1) $s_*(w \equiv w') = (sw \equiv sw')$ for $s \colon W(X) \to W(Y)$, $w, w' \in W(X)$ are of the same sort.

4.2) $s^x_w a \wedge (w \equiv w') < s^x_{w'} a$ for an arbitrary $a \in H(X), x \in X, w, w'$ of the same sort with x in $W(X)$, and $s^x_w \colon W(X) \to W(X)$ is defined by the rule: $s^x_w(x) = w, sy = y, y \in X,\ \ y \neq x$.

This completes the definition of the Halmos category for a given Θ.

2.3 The example $\mathrm{Hal}_\Theta(G)$**.**

Fix an algebra G in the variety Θ. Define the Halmos category $\mathrm{Hal}_\Theta(G)$ for the given G. Take a finite set X and consider the space $\mathrm{Hom}(W(X), G)$. We

have defined the action of quantifiers $\exists x$ for all $x \in X$ in the Boolean algebra $\mathrm{Bool}(W(X), G)$. The equality $w \equiv w'$ in $\mathrm{Bool}(W(X), G)$ is defined as a diagonal, coinciding with the set of all $\mu : W(X) \to G$ for which $w^\mu = w'^\mu$ holds. It is easy to check that in this case the algebra $\mathrm{Bool}(W(X), G)$ turns out to be a quantifier X-algebra with equalities. We set

$$\mathrm{Hal}_\Theta(G)(X) = \mathrm{Bool}(W(X), G).$$

Let now $s \colon W(X) \to W(Y)$ be given in Θ^0. We have:

$$\tilde{s} \colon \mathrm{Hom}(W(Y), G) \to \mathrm{Hom}(W(X), G)$$

defined by $\tilde{s}(\nu) = \nu s$ for any $\nu \colon W(Y) \to G$.

Now, if A is a subset in $\mathrm{Hom}(W(X), G)$, then $\nu \in s_* A = sA$ if and only if $\tilde{s}(\nu) = \nu s \in A$ We have a mapping:

$$s_* \colon \mathrm{Bool}(W(X), G) \to \mathrm{Bool}(W(Y), G)$$

which is a Boolean homomorphism. One can also check that s_* satisfies the conditions 3–4, thereby defining the Halmos category $\mathrm{Hal}_\Theta(G)$.

Note that a conjugate mapping

$$s^* \colon \mathrm{Bool}(W(Y), G) \to \mathrm{Bool}(W(X), G),$$

where the set $s^* B$ is the $\tilde{s}$-image of the set B for every $B \subset \mathrm{Hom}(W(Y), G)$ corresponds to each s_* . Here, s^* is not a Boolean homomorphism, but it preserves sums and zero.

It may be seen that such a conjugate mapping can be defined in any Halmos category. See, for example [Pl1].

2.4 Multi-sorted Halmos algebras.

Fix some infinite set $X^0 = (X_i^0, i \in \Gamma)$ and let Γ^0 be the set of all finite subsets $X = (X_i, i \in \Gamma)$ in X^0. In this section multi-sorted algebra means Γ^0-sorted. Every such algebra is of the form $H = (H(X), X \in \Gamma^0)$.

A few words about the signature of the algebras to be constructed. First, the signature includes L_X for every X together with equalities $w \equiv w', w, w'$ of the same sort in $W(X)$. The equalities are considered as nullary operations.

This is the signature in $H(X)$. Second, we consider symbols of operations of the type $s\colon W(X) \to W(Y)$. To each such symbol corresponds an unary operation $s\colon H(X) \to H(Y)$. Denote the signature consisting of all L_X, all equalities, and all $s\colon W(X) \to W(Y)$ by L_Θ. This is the *signature* of FOL in Θ in the *multi-sorted variant.*

Consider further the variety of Γ^0-sorted L_Θ-algebras, denoted by Hal_Θ. The identities of this variety exactly copy the definition of Halmos category. We call algebras from Hal_Θ *multi-sorted Halmos algebras.*

Every such algebra can be considered as a small Halmos category and vice versa. Thus we come from algebra to category and back without a special explanation.

2.5 Algebras of formulas.

First consider a multi-sorted set of atomic formulas $M = (M(X), X \in \Gamma^0)$, with $M(X) = M_X$ defined as above. All $w \equiv w'$ are viewed as symbols of nullary operations-equalities. The set of symbols of relations Φ is fixed.

Denote by $H_{\Phi\Theta} = (H_{\Phi\Theta}(X), X \in \Gamma^0)$ the absolutely free L_Θ-algebra over the set M. This is the algebra of formulas of pure FOL in the given Θ.

Now denote by $\tilde{H}_{\Phi\Theta}$ the result of factorization of the algebra $H_{\Phi\Theta}$ by the identities of the variety Hal_Θ. It is the free Halmos algebra over the set of atomic formulas M.

Let us introduce the following defining relations:

$$(*) \qquad s_*\varphi(w_1, \dots, w_n) = \varphi(sw_1, \dots, sw_n)$$

for all $s\colon W(X) \to W(Y)$ and all formulas of the type $\varphi(w_1, \dots, w_n)$ in $M(X)$.

In the sequel the principal role will play the Halmos algebra $\mathrm{Hal}_\Theta(\Phi) = \mathrm{Hal}_{\Phi\Theta}$, defined as a quotient algebra of the free algebra $\tilde{H}_{\Phi\Theta}$ by the relations of the (*) type. Elements of this algebra are defined to be *compressed formulas.*

Consider now values of formulas. First of all take a mapping

$$\mathrm{Val}_f = (\mathrm{Val}_f^X, X \in \Gamma^0)\colon\ M \to \mathrm{Hal}_\Theta(G).$$

For the model (G, Φ, f) the mapping $\mathrm{Val}_f^X\colon M_X \to \mathrm{Bool}(W(X), G) = \mathrm{Hal}_\Theta(G)(X)$ has been defined.

This mapping is uniquely extended up to the homomorphisms

$$\mathrm{Val}_f\colon H_{\Phi\Theta} \to \mathrm{Hal}_\Theta(G),$$

$$\mathrm{Val}_f\colon \tilde{H}_{\Phi\Theta} \to \mathrm{Hal}_\Theta(G).$$

Note that the relations $(*)$ hold in every algebra $\mathrm{Hal}_\Theta(G)$ and this gives a canonical homomorphism of Halmos algebras

$$Val_f : \mathrm{Hal}_\Theta(\Phi) \to \mathrm{Hal}_\Theta(G).$$

It determines the value of the formulas $\mathrm{Val}_f(u)$ (pure and compressed) in the given model (G, Φ, f).

We call two pure formulas u and v of the given type X *semantically equivalent*, if $\mathrm{Val}_f(u) = \mathrm{Val}_f(v)$ for every model (G, Φ, f).

The following main theorem takes place [Pl2]:

Theorem 1. *Two formulas u and v are semantically equivalent if and only if the corresponding compressed formulas $\overline{u}$ and $\overline{v}$ coincide in the algebra* $\mathrm{Hal}_\Theta(\Phi)$.

This theorem explains the role of algebra $\mathrm{Hal}_\Theta(\Phi)$ as a main structure of the multi-sorted algebraic logic for FOL in the given Θ. The same algebra plays an essential part in the algebraic geometry in the FOL in Θ. In particular, the role of the algebras $\mathrm{Hal}_\Theta(G)$ is underlined by the following theorem [Pl2]:

Theorem 2. *The algebras* $\mathrm{Hal}_\Theta(G)$ *over different $G \in \Theta$ generate the variety of Halmos algebras* Hal_Θ.

Define the notion of the logical kernel of a homomorphism.

Let the homomorphism $\mu\colon W(X) \to G$ be given. One can view its kernel $\mathrm{Ker}\mu$ as a system of all formulas $w \equiv w'$ with w, w' of the same sort in $W(X)$, for which $\mu \in \mathrm{Val}(w \equiv w')$.

Logical kernel $\mathrm{LogKer}\mu$ naturally generates the standard $\mathrm{Ker}\mu$. We set: the formula $u \in \mathrm{Hal}_{\Phi\Theta}(X)$ belongs to $\mathrm{LogKer}(\mu)$ if the point μ lies in $\mathrm{Val}_f(u)$, i.e., if μ is a solution of the "equation" u in the given model (G, Φ, f). It is easy to understand, that for every point μ its logical kernel is an ultrafilter of the Boolean algebra $\mathrm{Hal}_{\Phi\Theta}(X)$. It is also clear, that the kernel $\mathrm{Ker}\mu$ is the set of all equalities in the logical kernel.

3. Categories of algebraic sets

3.1 Preliminary remarks.

We defined in the subsection 1.3 the algebraic sets determined by FOL formulas. Now we work with the compressed formulas, i.e., the formulas of the algebra $\mathrm{Hal}_\Theta(\Phi) = \mathrm{Hal}_{\Phi\Theta}$. Correspondingly, we have to extend the definition of Galois correspondence from 1.3 to the case of compressed formulas, i.e, to the elements of $\mathrm{Hal}_\Theta(\Phi)$.

For the given place X consider sets of formulas T in $\mathrm{Hal}_{\Phi\Theta}(X)$ and the sets of points A in the space $\mathrm{Hom}(W(X), G)$. Having the model (G, Φ, f), we establish a Galois correspondence between sets of elements (compressed formulas) in the Halmos algebra $\mathrm{Hal}_\Theta(\Phi)$ and sets of points in the space $\mathrm{Hom}(W, G)$: :

$$T^f = A = \bigcap_{u \in T} \mathrm{Val}_f(u) = \{\mu | T \subset \mathrm{LogKer}(\mu)\}$$

$$A^f = T = \{u | A \subset \mathrm{Val}_f(u)\} = \bigcap_{\mu \in A} \mathrm{LogKer}(\mu).$$

As in 1.3, we call a set A represented as $A = T^f$ an *algebraic set* or *algebraic variety* for the given model (G, Φ, f).

The set T, represented as $A^f = T$, is always a filter of the Boolean algebra $\mathrm{Hal}_{\Phi\Theta}(X)$, since by definition it is an intersection of ultrafilters. We call it an *f-closed filter*. One can consider a Boolean algebra $\mathrm{Hal}_{\Phi\Theta}(X)/T$ for this T. If $T^f = A$ and $A^f = T$, then the algebra $\mathrm{Hal}_{\Phi\Theta}(X)/T$ is considered as an invariant of the algebraic set A. This invariant is a *coordinate algebra* of the set A. It can be viewed as an algebra of regular functions determined on the variety A (see [Pl2]).

Suppose an algebraic set A is given. A filter $T = A^f$ can be treated as the theory of a the set A for the fixed model (G, Φ, f).

Every algebraic set, defined in Subsection 1.3, is also an algebraic set according to this new definition. The opposite is not true, because in the new variant additional operations of the type $s \colon W(X) \to W(Y)$ are involved in the formulas.

We will return later to the structure of algebraic sets.

Consider now the relation between the Galois correspondence and morphisms of Halmos categories.

For every $s \colon W(X) \to W(Y)$ and every A of the type X we considered a set $B = s_* A$ of the type Y. If B is of the type Y, then $A = s^* B$ is of the type X.

Define the operations s_* and s^* on the sets of formulas.

If T is a set of formulas in $\mathrm{Hal}_{\Phi\Theta}(Y)$, then s_*T is a set of formulas in $\mathrm{Hal}_{\Phi\Theta}(X)$ defined by the rule:

$$u \in s_*T \Leftrightarrow su \in T.$$

If T is a set of formulas in $\mathrm{Hal}_{\Phi\Theta}(X)$, then s^*T is contained in $\mathrm{Hal}_{\Phi\Theta}(Y)$ and it is defined by

$$u \in s^*T \ \text{ if } \quad u = sv, \quad v \in T.$$

The following theorem [Pl2] holds:

Theorem 3.

1. If T lies in $\mathrm{Hal}_{\Phi\Theta}(X)$, then

$$(s^*T)^f = s_*T^f = sT^f.$$

2. If $B \subset \mathrm{Hom}(W(Y), G)$, then

$$(s^*B)^f = s_*B^f.$$

3. If $A \subset \mathrm{Hom}(W(X), G)$, then $s^*A^f \subset (s_*A)^f$.

It follows from these rules that

1. If $A = T^f$ is an algebraic set, then sA is also an algebraic set.

2. If $T = B^f$ is f-closed, then $sT = s_*T$ is f-closed.

3.2. Categories $K_{\Phi\Theta}(f)$ and $C_{\Phi\Theta}(f)$.

Fix a model (G, Φ, f) and define a category of algebraic sets $K_{\Phi\Theta}(f)$ for this model. Objects of this category have the form (X, A), where $A = T^f$ for some T. X is the place for both A and T.

Let us now define morphisms $(X, A) \to (Y, B)$. For $s\colon W(Y) \to W(X)$ we say that s is *admissible* for A and B if $\tilde{s}(\nu) = \nu s \in B$ for any $\nu \in A$. It is clear that s is admissible for A and B if $A \subset sB$. A mapping $[s] : A \to B$ corresponds to each s admissible for A and B. Note that for the equal $[s_1]$ and $[s_2]$ the corresponding $\tilde{s}_1$ and $\tilde{s}_2$ can be different.

We consider *weak* and *exact* categories $K_{\Phi\Theta}(f)$. In the first one the morphisms are of the form $\tilde{s} : (X, A) \to (Y, B)$, while in the second one they are of the form $[s] : (X, A) \to (Y, B)$. Here, s assumed to be admissible for A and B.

If s_1 is admissible for A and B and s_2 for B and C, then $A \subset s_1 B$, $B \subset s_2 C$, $s_1 B \subset s_1 s_2 C$, and $s_1 s_2$ is admissible for A and C.

Define now a category $C_{\Phi\Theta}(f)$. Its objects are Boolean algebras of the type $\mathrm{Hal}_{\Phi\Theta}(X)/T$, where $T = A^f$ for some A.

Consider morphisms

$$\mathrm{Hal}_{\Phi\Theta}(Y)/T_2 \xrightarrow{\overline{s}} \mathrm{Hal}_{\Phi\Theta}(X)/T_1.$$

We proceed here from $s\colon W(Y) \to W(X)$ and pass to the new $s : \mathrm{Hal}_{\Phi\Theta}(Y) \to \mathrm{Hal}_{\Phi\Theta}(X)$. Assume that $su \in T_1$ for every $u \in T_2$. The homomorphism s is admissible for T_2 and T_1 in this sense. Define homomorphisms $\overline{s}$ for such s. This defines morphisms in $C_{\Phi\Theta}(f)$.

The next two straightforward propositions determine the correspondence between the categories $K_{\Phi\Theta}(f)$ and $C_{\Phi\Theta}(f)$.

Proposition 1. *A homomorphism $s\colon W(Y) \to W(X)$ is admissible for the sets (X, A) and (Y, B) if and only if it is admissible for $T_2 = B^f$ and $T_1 = A^f$.*

Proposition 2. *If $s_1, s_2\colon W(Y) \to W(X)$ are admissible for A and B, then $[s_1] = [s_2]$ implies $\overline{s_1} = \overline{s_2}$.*

It follows from these two propositions that the transition

$$(X, A) \to \mathrm{Hal}_{\Phi\Theta}(X)/A^f$$

determines a contravariant functor

$$K_{\Phi\Theta}(f) \to C_{\Phi\Theta}(f)$$

for weak and exact categories $K_{\Phi\Theta}(f)$. Duality for these categories takes place under some additional conditions.

3.3 Categories $K_{\Phi\Theta}$ and $C_{\Phi\Theta}$.

In the categories $K_{\Phi\Theta}$ and $C_{\Phi\Theta}$ the model (G, Φ, f) is not fixed. Objects of $K_{\Phi\Theta}$ have the form $(X, A; G, f)$. Here f is a interpretation of the set Φ in the algebra G, fixed for the category $K_{\Phi\Theta}$, and $A = T^f$ for some $T \subset \mathrm{Hal}_{\Phi\Theta}(X)$.

Define morphisms

$$(X, A; G_1, f_1) \to (Y, B; G_2, f_2)$$

They act on all components of the objects. Proceed from the commutative diagram

$$\begin{tikzcd} W(Y) \arrow[r, "s"] \arrow[d, "\nu'"'] & W(X) \arrow[d, "\nu"] \\ G_2 & G_1 \arrow[l, "\delta"'] \end{tikzcd}$$

Consider a pair (s, δ) and write $(s, \delta)(\nu) = \nu' = \delta\nu s$.

Let now $A = T_1^{f_1}$ be of the type X and $B = T_2^{f_2}$ of the type Y. We say that the pair (s, δ) is *admissible* for A and B if $(s, \delta)(\nu) \in B$ for every $\nu \in A$.

We need some further auxiliary remarks. For every $\delta : G_1 \to G_2$ and every X we have a mapping

$$\tilde{\delta} : \mathrm{Hom}(W(X), G_1) \to \mathrm{Hom}(W(X), G_2)$$

defined by the rule

$$\tilde{\delta}(\nu) = \delta\nu, \nu \in \mathrm{Hom}(W(X), G_1).$$

Define $\delta_* A \subset \mathrm{Hom}(W(X), G_1)$ for every $A \subset \mathrm{Hom}(W(X), G_2)$, by setting

$$\nu \in \delta_* A \ \text{ if } \ \ \delta\nu = \tilde{\delta}(\nu) \in A.$$

We write also $\delta_* A = \delta A$, and consider δ^* determined by: if $A \subset \mathrm{Hom}(W(X), G_1)$, then $\delta^* A \subset \mathrm{Hom}(W(X), G_2)$ and $\nu \in \delta^* A$ if $\nu = \delta\nu_1, \nu_1 \in A$.

Now we can say that the pair (s, δ) is admissible for A and B if $\delta^* A \subset sB$, or, the same, $A \subset \delta s B = s\delta B$.

We have morphisms

$$(s, \delta) : (X, A; G_1, f_1) \to (Y, B; G_2, f_2)$$

and

$$([s], \delta)\colon (X, A; G_1, f_1) \to (Y, B; G_2, f_2)$$

for the admissible (s, δ). Here $[s] : A \to B$ is a mapping, induced by the pair (s, δ). We get weak and exact categories $K_{\Phi\Theta}$. It can be proven that the pair (s, δ) is admissible for A and B if and only if the homomorphism $s\colon \mathrm{Hal}_{\Phi\Theta}(Y) \to Hal_{\Phi\Theta}(X)$ is admissible in respect to $T_2 = B^{f_2}$ and $T_1 = (\delta^* A)^{f_2}$. This leads to a natural definition of the category $C_{\Phi\Theta}$ with contravariant functor $K_{\Phi\Theta} \to C_{\Phi\Theta}$.

Let us define the categories $K_{\Phi\Theta}(G)$ and $C_{\Phi\Theta}(G)$. Here G is a fixed algebra in Θ, while the interpretations f of the set Φ in G change.

The objects in $K_{\Phi\Theta}(G)$ have the form

$$(X, A; f).$$

The morphisms

$$(X, A, f_1) \to (Y, B, f_2)$$

are defined according to the general definition of the morphisms in $K_{\Phi\Theta}$ with identical $\delta = \varepsilon \colon G \to G$.

Objects in $C_{\Phi\Theta}(G)$ have the form

$$(\mathrm{Hal}_{\Phi\Theta}(X)/T, f), \ \text{ where } \ T = A^f$$

for some A of the type X.

The transition

$$(X, A; f) \to (\mathrm{Hal}_{\Phi\Theta}(X)/A^f, f)$$

determines the functor $K_{\Phi\Theta}(G) \to C_{\Phi\Theta}(G)$. Here $K_{\Phi\Theta}(G)$ is a subcategory in $K_{\Phi\Theta}$ and every $K_{\Phi\Theta}(f)$ is a subcategory in $K_{\Phi\Theta}(G)$. The same holds for C. See also [NP].

4. Categories of elementary knowledge

4.1 The category $\mathbf{Know}_{\Phi\Theta}(f)$.

In Subsection 1.3 we defined knowledge as

$$(X, T, A, (G, \Phi, f)),$$

where each component has the corresponding meaning. Fix a model (subject of knowledge) (G, Φ, f). Let us define a category of knowledge for this model and denote it by $\mathrm{Know}_{\Phi\Theta}(f)$. This is the knowledge category for the given subject of knowledge. Since the model is fixed, the objects of the category $\mathrm{Know}_{\Phi\Theta}(f)$ have to have the form (X, T, A). We do not fix the subject of knowledge in the notation of the object, since it is fixed in the notation of the category.

The set X is multi-sorted. It marks the "place" where the knowledge is situated. The set X points also the "place of the knowledge", i.e., the space of the knowledge $Hom(W(X), G)$, while the subject of the knowledge (G, Φ, f) is given. The set

T is the description of the knowledge in the algebra $Hal_\Theta(X)$, and $A = T^f$ is the content of knowledge, depending on T and f. The set $T^{ff} = A^f$ is the full description of the knowledge (X, T, A) which is a Boolean filter in $\mathrm{Hal}_{\Phi\Theta}(X)$.

Now about morphisms $(X, T_1, A) \to (Y, T_2, B)$. Take $s\colon W(Y) \to W(X)$. We have also $s\colon \mathrm{Hal}_{\Phi\Theta}(Y) \to \mathrm{Hal}_{\Phi\Theta}(X)$ (see 2.2). This is a homomorphism of Boolean algebras. The homomorphism s gives rise to

$$\tilde{s}\colon Hom(W(X), G) \to Hom(W(Y), G).$$

As above, the first s is admissible for A and B if $\tilde{s}(\nu) = \nu s \in B$ for every point $\nu\colon W(X) \to G$ in A.

As we know, s is admissible for A and B if and only if $su \in A^f$ for every $u \in B^f$. This holds for s_*, for which we have also a homomorphism $\overline{s} : \mathrm{Hal}_{\Phi\Theta}(Y)/B^f \to \mathrm{Hal}_{\Phi\Theta}(X)/A^f$. It is easy to prove that s is admissible for A and B if and only if $su \in A^f$ holds for every $u \in T_2$. We consider admissible s as a morphism

$$s\colon (X, T_1, A) \to (Y, T_2, B),$$

in the weak category $\mathrm{Know}_{\Phi\Theta}(f)$.

We have $\tilde{s}(\nu) = \nu s \in B$ if $\nu \in A$, and s induces a mapping $[s]\colon A \to B$. Simultaneously, there is a mapping $s\colon T_2 \to A^f$ and a homomorphism

$$\overline{s}\colon \mathrm{Hal}_{\Phi\Theta}(Y)/B^f \to \mathrm{Hal}_{\Phi\Theta}(X)/A^f.$$

We have already mentioned (Proposition 2) that $\overline{s}_1 = \overline{s}_2$ follows from $[s_1] = [s_2]$. Thus, we can take the morphisms of the form

$$[s]\colon (X, T_1, A) \to (Y, T_2, B),$$

for the morphisms of the exact category $\mathrm{Know}_{\Phi\Theta}(f)$. The canonical functors $\mathrm{Know}_{\Phi\Theta}(f) \to K_{\Phi\Theta}(f)$ for weak and exact categories are given by the transition $(X, T, A) \to (X, A)$. In this transition we "forget" to fix the description of knowledge T.

4.2 The category $\mathrm{Know}_{\Phi\Theta}$.

Let us define the category of elementary knowledge for the whole applied field $\Phi\Theta$; the subject of the knowledge (G, Φ, f) is not fixed. As earlier, we proceed

from the category $\Phi\Theta$ whose morphisms are homomorphisms in Θ. They ignore the relations from Φ.

An object of the knowledge category $\mathrm{Know}_{\Phi\Theta}$ has the form

$$(X, T, A; (G, \Phi, f)),$$

and we write $(X, T, A; G, f)$, because Φ is fixed for the category. Here X marks the place of knowledge. The components $A = T^f$, G and f may change.

Consider morphisms:

$$(X, T_1, A; G_1, f_1) \to (Y, T_2, B; G_2, f_2).$$

We apply the same approach as in Section 3.3 with some modifications.

Start from $s : W(Y) \to W(X)$ and $\delta : G_1 \to G_2$. These s and δ should correlate. Let us explain the correlation condition. Take a set $A_1 = \{\delta\nu, \nu \in A\} = \delta^* A$ and take further $T_1^\delta = A_1^{f_2}$. Correlation of s and δ means that $su \in T_1^\delta$ holds for any $u \in T_2$. The same holds for every $u \in B^{f_2}$. The last also says that there is a homomorphism

$$\overline{s}\colon \mathrm{Hal}_{\Phi\Theta}(Y)/B^{f_2} \to \mathrm{Hal}_{\Phi\Theta}(X)/A_1^{f_2}.$$

The first of the two mappings $(s, \delta)\colon A \to B$ and $s\colon T_2 \to T_1^\delta$ transforms the content of knowledge, while the second one acts on the description. Here T_2 and T_1^δ describe knowledge associated with the same subject (G_2, Φ, f_2).

With the fixed δ there is also an exact mapping $([s], \delta) : A \to B$. This brings us to weak and exact categories $\mathrm{Know}_{\Phi\Theta}$. The morphisms of the first one are (s, δ) and in the second one they are of the form $([s], \delta)$ for $(X, T_1, A; G_2, f_1) \to (Y, T_2, B; G_2, f_2)$. The canonical functors $\mathrm{Know}_{\Phi\Theta} \to K_{\Phi\Theta}$ are defined by the transition

$$(X, T, A; G, f) \to (X, A; G, f).$$

As above, we remove the description of knowledge from the notations.

4.3 Categories $K_{\Phi\Theta}(G)$ and $\mathrm{Know}_{\Phi\Theta}(G)$.

An algebra $G \in \Theta$ is fixed in the categories $K_{\Phi\Theta}(G)$ and $\mathrm{Know}_{\Phi\Theta}(G)$. A set of symbols of relations Φ is fixed as usual, but interpretations f of Φ in G may change. Thus, $K_{\Phi\Theta}(G)$ is a subcategory in $K_{\Phi\Theta}$ and $\mathrm{Know}_{\Phi\Theta}(G)$ is a subcategory in $\mathrm{Know}_{\Phi\Theta}$. Here the corresponding $\delta : G \to G$ are identical homomorphisms.

Objects of the category $K_{\Phi\Theta}(G)$ have the form (X, A, f), and those of the category $\mathrm{Know}_{\Phi\Theta}(G)$ are written as (X, T, A, f). There is a canonical functor $\mathrm{Know}_{\Phi\Theta}(G) \to K_{\Phi\Theta}(G)$. As for morphisms

$$(X, A, f_1) \to (Y, B, f_2) \text{ and}$$
$$(X, T_1, A, f_1) \to (Y, T_2, B, f_2),$$

we note that $A = A_1, A_1^{f_2} = T_1^{\delta}$ and $A^{f_2} = T_1^{f_1 f_2}$. Hence, the corresponding admissible $s : W(Y) \to W(X)$ transfers each $u \in T_2$ into $su \in T_1^{f_1 f_2}$ and it induces a homomorphism

$$\bar{s} : \mathrm{Hal}_{\Phi\Theta}(Y)/B^{f_2} \to \mathrm{Hal}_{\Phi\Theta}(X)/A^{f_2}.$$

Every s gives a mapping $[s] : A \to B$. This defines a morphism $(X, A, f_1) \to (Y, B, f_2)$.

5. Knowledge bases

5.1.Category of knowledge description.

Denote the category of knowledge description by $L_{\Phi\Theta}$ or $L_{\Theta}(\Phi)$.

Its objects are of the form (X, T), where X is a finite set and T is a set of formulas of $\mathrm{Hal}_{\Phi\Theta}(X)$. Define morphisms $(X, T_1) \to (X, T_2)$. According to the definition of the category $\mathrm{Hal}_{\Theta}(\Phi)$ proceed from the functor $\Theta^0 \to \mathrm{Hal}_{\Theta}(\Phi)$ which assigns a mapping $s_* : \mathrm{Hal}_{\Phi\Theta}(X) \to \mathrm{Hal}_{\Phi\Theta}(Y)$ to every homomorphism $s : W(X) \to W(Y)$. We say that s is *admissible in respect to T_1 and T_2* if $s_*(u) \in T_2$ for every $u \in T_1$. For such admissible s we have a mapping $s_* : T_1 \to T_2$ which determines

$$s_* : (X, T_1) \to (X, T_2).$$

5.2 Functor of transition from knowledge description to knowledge content.

Proceed from the model (G, Φ, f) and consider a functor

$$\mathrm{Ct}_f : L_{\Phi\Theta} \to K_{\Phi\Theta}(f).$$

Here, $K_{\Phi\Theta}(f)$ is the corresponding category of algebraic (elementary) sets over the given model and Ct stands for "contents". The functor Ct_f is a contravariant one. To every object (X, T) of the category $L_{\Phi\Theta}$ it assigns the corresponding content $(X, T^f) = (X, A)$ which is an object of the category $K_{\Phi\Theta}(f)$.

Now one has to define the functor Ct_f on morphisms. Let a morphism

$$s_* : (Y, T_2) \to (X, T_1)$$

be given for $s : W(Y) \to W(X)$. Show that s induces a morphism

$$\widetilde{s_*} : (X, A) \to (Y, B),$$

where $A = T_1^f$, and $B = T_2^f$.

We proceed from $\widetilde{s} : \mathrm{Hom}(W(X), G) \to \mathrm{Hom}(W(Y), G)$.

Let us define a transition $s \to \widetilde{s}$.

Check first that if s is admissible for T_2 and T_1 then this s is admissible for $A = T_1^f$ and $B = T_2^f$. The last means that $\widetilde{s}(\nu) \in B$ if $\nu \in A$. The inclusion $\nu \in A$ says that $\nu \in \mathrm{Val}_f(v)$ for every $v \in T_1$. We need to verify that $\nu s \in B$, that is $\nu s \in \mathrm{Val}_f(u)$ for every $u \in T_2$.

Take an arbitrary $u \in T_2$. We have: $v = s_*(u) \in T_1$; $\nu \in \mathrm{Val}_f(v) = \mathrm{Val}_f(s_* u) = s\mathrm{Val}_f(u)$. This gives $\nu s \in \mathrm{Val}_f(u)$. We used that s and Val_f commute, since Val_f is a homomorphism of algebras.

The mapping $[s] : A \to B$ corresponds to the homomorphism $s : W(Y) \to W(X)$. This mapping is considered simultaneously as a morphism in the category $K_{\Phi\Theta}(f)$ (see 3.2)

$$[s] : (X, A) \to (Y, B).$$

We define: $\mathrm{Ct}_f(s_*) = \widetilde{s}_* = [s]$.

Check now compatibility of the definition of Ct_f with the multiplication of morphisms. Given $s_1 : W(X) \to W(Y)$ and $s_2 : W(Y) \to W(Z)$ we have $s_2 s_1 : W(X) \to W(Z)$. Using the fact that the transition $\Theta^0 \to \mathrm{Hal}_\Theta(\Phi)$ is a functor, we get $(s_2 s_1)_* = s_{2*} s_{1*}$. Here, we have

$$s_{1*} : \mathrm{Hal}_{\Phi\Theta}(X) \to \mathrm{Hal}_{\Phi\Theta}(Y),$$

$$s_{2*} : \mathrm{Hal}_{\Phi\Theta}(Y) \to \mathrm{Hal}_{\Phi\Theta}(Z),$$

and

$$(s_2 s_1)_* : \mathrm{Hal}_{\Phi\Theta}(X) \to \mathrm{Hal}_{\Phi\Theta}(Z)$$

Let (X, T_1), (Y, T_2) and (Z, T_3) be objects in $L_\Theta(\Phi)$, and s_1, s_2 admissible in respect to T_1, T_2 and, correspondingly, for T_2, T_3. In this case there are morphisms

$$s_{1*} : (X, T_1) \to (Y, T_2),$$

$$s_{2*} : (Y, T_2) \to (Z, T_3),$$

and

$$s_{2*}s_{1*} = (s_2 s_1)_* : (X, T_1) \to (Z, T_3).$$

Take $T_1^f = A$, $T_2^f = B$, $T_3^f = C$. We have

$$\widetilde{s_{1*}} : (Y, B) \to (X, A),$$

$$\widetilde{s_{2*}} : (Z, C) \to (Y, B),$$

and

$$\widetilde{s_2 s_1}_* = \widetilde{s_{1*}}\widetilde{s_{2*}} : (Z, C) \to (X, A).$$

This gives compatibility of the functor Ct_f with the multiplication of morphisms. Compatibility with the unity morphism is evident. This finishes the definition of the contravariant functor $\mathrm{Ct}_f : L_{\Phi\Theta} \to K_{\Phi\Theta}(f)$.

5.3 Homomorphisms of Halmos algebras $\mathrm{Hal}_\Theta(\Phi)$ and functors of the categories $L_\Theta(\Phi)$.

Given a homomorphism $\beta : \mathrm{Hal}_\Theta(\Phi_1) \to \mathrm{Hal}_\Theta(\Phi_2)$, define the corresponding functor $\widetilde{\beta} : L_\Theta(\Phi_1) \to L_\Theta(\Phi_2)$. For every set of formulas $T \subset \mathrm{Hal}_{\Phi_1\Theta}(X)$, denote by T^β the set $T^\beta = \{u^\beta, u \in T\}$. If (X, T) is an object in $L_\Theta(\Phi_1)$, then, setting

$$\widetilde{\beta}(X, T) = (X, T^\beta),$$

we get an object in $L_\Theta(\Phi_2)$.

In order to define the functor $\widetilde{\beta}$ on morphisms let us make a remark. Proceed from the functors $\Theta^0 \to \mathrm{Hal}_\Theta(\Phi_1)$ and $\Theta^0 \to \mathrm{Hal}_\Theta(\Phi_2)$. The morphisms

$$s_*^1 : \mathrm{Hal}_{\Phi_1\Theta}(X) \to \mathrm{Hal}_{\Phi_1\Theta}(Y),$$

$$s_*^2 : \mathrm{Hal}_{\Phi_2\Theta}(X) \to \mathrm{Hal}_{\Phi_2\Theta}(Y)$$

correspond to every $s : W(X) \to W(Y)$. We have also

$$\beta = (\beta_X, X \in \Gamma^0) : \mathrm{Hal}_\Theta(\Phi_1) \to \mathrm{Hal}_\Theta(\Phi_2).$$

The fact that the homomorphism β is compatible with the operation s is represented by the commutative diagram

$$\begin{tikzcd} \mathrm{Hal}_{\Phi_1\Theta}(X) \arrow[r, "s^1_*"] \arrow[d, "\beta_X"'] & \mathrm{Hal}_{\Phi_1\Theta}(Y) \arrow[d, "\beta_Y"] \\ \mathrm{Hal}_{\Phi_2\Theta}(X) \arrow[r, "s^2_*"] & \mathrm{Hal}_{\Phi_2\Theta}(Y) \end{tikzcd}$$

So, for a homomorphism $s : W(X) \to W(Y)$ we have the equality $\beta_Y s^1_*(u) = s^2_* \beta_X(u)$ for every $u \in \mathrm{Hal}_{\Phi_1\Theta}(X)$.

Now we are able to define an action of the functor $\widetilde{\beta}$ on morphisms. Let a morphism $s^1_* : (X, T_1) \to (Y, T_2)$ in the category $L_{\Phi_1\Theta}$ be given and $s^1_*(u) \in T_2$ if $u \in T_1$. Then, we have $s^2_*(v) \in T_2^\beta$ if $v \in T_1^\beta$.

Indeed, let $v = \beta_X(u)$, $u \in T_1$, $v \in T_1^{\beta_X}$. We have:

$$s^2_* \beta_X(u) = s^2_*(v) = \beta_Y s^1_*(u) \in T_2^{\beta_Y},$$

since $s^1_*(u) \in T_2$. Hence, $s^2_*(v) \in T_2^{\beta_Y}$ for every $v = \beta_X(u) \in T_1^{\beta_X}$.

We set $s^2_* = \widetilde{\beta}(s^1_*) : T_1^{\beta_X} \to T_2^{\beta_Y}$. A morphism

$$s^2_* = \widetilde{\beta}(s^1_*) : (X, T_1^{\beta_X}) \to (Y, T_2^{\beta_Y})$$

corresponds to $s^1_* : (X, T_1) \to (Y, T_2)$.

Check now compatibility of the transition $s^1_* \to s^2_*$ with the multiplication of morphisms. Given $s_1 : W(X) \to W(Y)$ and $s_2 : W(Y) \to W(Z)$, we have $s_2 s_1 : W(X) \to W(Z)$. Using once more the fact that the transition $\Theta^0 \to \mathrm{Hal}_\Theta(\Phi)$ is a functor, we get

$$(s^1_2 s^1_1)_* = s^1_{2*} s^1_{1*},$$

$$(s^2_2 s^2_1)_* = s^2_{2*} s^2_{1*},$$

Apply $\widetilde{\beta}$. We need to verify that $\widetilde{\beta}(s^1_{2*} s^1_{1*}) = \widetilde{\beta}(s^1_{2*}) \widetilde{\beta}(s^1_{1*})$. We have

$$\widetilde{\beta}(s^1_{2*} s^1_{1*}) = \widetilde{\beta}(s^1_2 s^1_1)_* = (s^2_2 s^2_1)_* = s^2_{2*} s^2_{1*} = \widetilde{\beta}(s^1_{2*}) \widetilde{\beta}(s^1_{1*}).$$

This gives compatibility with the multiplication as well as with the unit. Hence, we have the functor $\widetilde{\beta} : L_\Theta(\Phi_1) \to L_\Theta(\Phi_2)$.

5.4 Knowledge bases.

We proceed from a multi-model (G, Φ, F). A multi-model (G, Φ, F) defines a system of models $(G, \Phi, f,)$ where f runs the set F. Here G is an algebra in Θ, and Φ is a set of relations. Recall that both the algebra $G \in \Theta$ and a relation $f \in F$ are multi-sorted. The set F is a set of instances f, where f is a interpretation of the set Φ in G.

To every such multi-model corresponds a knowledge base $KB = KB(G, \Phi, F)$. The definition slightly differs from that of [PTP].

Definition. *A knowledge base $KB = KB(G, \Phi, F)$ consists of two categories. The first one is the category of knowledge description $L_\Theta(\Phi)$, and the second one is the category of knowledge content $K_{\Phi\Theta}(f)$. These two categories are related by the functor*

$$\mathrm{Ct}_f : L_\Theta(\Phi) \to K_{\Phi\Theta}(f).$$

This functor Ct_f transforms knowledge description to content of knowledge. We do not assume that between different f_1 and f_2 in F there are any ties: instances are independent. On the other hand, between some f_1 and f_2 there may be relations that we will try to take into account (see Section 7).

A content of knowledge $\mathrm{Ct}_f(X, T) = (X, T^f)$ corresponds to an object (X, T) of the category $L_\Theta(\Phi)$, which is a description of knowledge. We view the description T as a *query* to a knowledge base, and $A = T^f$ as a *reply to this query.*

Besides, if there is a relation s_* between (X, T_1) and (Y, T_2), then there will be a relation $\widetilde{s} = \widetilde{s_*}$ between (X, A) and (Y, B), where $A = T_1^f$, $B = T_2^f$.

This peculiarity of the definition naturally reflects geometrical essence of knowledge.

In fact, in this definition of a knowledge base the category of knowledge is decomposed to two categories: the category of description of knowledge and the category of content of knowledge, tied by the functor of transition from description to content.

6. Equivalence of knowledge bases

6.1 Definition.

Let the knowledge bases $KB_1 = KB(G_1, \Phi_1, F_1)$ and $KB_2 = KB(G_2, \Phi_2, F_2)$ correspond to the given multi-models (G_1, Φ_1, F_1) and (G_2, Φ_2, F_2).

Definition 1. *Knowledge bases KB_1 and KB_2 are called informationally equivalent, if there exists a bijection $\alpha : F_1 \to F_2$ such that for every $f \in F_1$ there exist homomorphisms*

$$\beta_f : \mathrm{Hal}_\Theta(\Phi_1) \to \mathrm{Hal}_\Theta(\Phi_2)$$

$$\beta'_f : \mathrm{Hal}_\Theta(\Phi_2) \to \mathrm{Hal}_\Theta(\Phi_1)$$

and an isomorphism of categories

$$\widetilde{\gamma}_f : K_{\Phi_1\Theta}(f) \to K_{\Phi_2\Theta}(f^\alpha)$$

such that the commutative diagrams of functors of categories hold:

$$\begin{tikzcd} L_\Theta(\Phi_1) \arrow[r, "\widetilde{\beta}_f"] \arrow[d, "\mathrm{Ct}_f"'] & L_\Theta(\Phi_2) \arrow[d, "\mathrm{Ct}_{f^\alpha}"] \\ K_{\Phi_1\Theta}(f) \arrow[r, "\widetilde{\gamma}_f"] & K_{\Phi_2\Theta}(f^\alpha) \end{tikzcd}$$

and

$$\begin{tikzcd} L_\Theta(\Phi_1) \arrow[d, "\mathrm{Ct}_f"'] & L_\Theta(\Phi_2) \arrow[l, "\widetilde{\beta}'_f"'] \arrow[d, "\mathrm{Ct}_{f^\alpha}"] \\ K_{\Phi_1\Theta}(f) & K_{\Phi_2\Theta}(f^\alpha) \arrow[l, "(\widetilde{\gamma}_f)^{-1}"'] \end{tikzcd}$$

Denote these diagrams by $*$ and $**$ respectively. Rewrite commutative diagrams for the object (X, T) of the category $L_\Theta(\Phi_1)$ in the form $(X, T^f)^{\widetilde{\gamma}_f} = (X, T^{\beta_f f^\alpha})$ and for the object (X, T) of the category $L_\Theta(\Phi_2)$ in the form $(X, T^{f^\alpha})^{\widetilde{\gamma}_f^{-1}} = (X, T^{\beta'_f f})$.

From this follows

$$(X, T^f) = (X, T^{\beta_f f^\alpha})^{\widetilde{(\gamma_f)}^{-1}},$$

$$(X, T^{f^\alpha}) = (X, T^{\beta'_f f})^{\widetilde{\gamma}_f}.$$

The last means that everything which can be known from KB_1 can be also known from KB_2 and vice versa. Similar property holds for morphisms, i.e. for relations between objects. Equivalence of knowledge bases we consider as a triple $(\alpha, *, **)$, where $\alpha : F_1 \to F_2$ is a bijection, while $*$ and $**$ define the corresponding diagrams for every $f \in F_1$.

The next proposition deals with the transition from knowledge bases to databases. Let B_Φ be the image of the homomorphism $\mathrm{Val}_f : \mathrm{Hal}_\Theta(\Phi) \to \mathrm{Hal}_\Theta(G)$

Proposition 3. *If a bijection $\alpha : F_1 \to F_2$ determines equivalence of the bases KB_1 and KB_2 then for every $f \in F_1$ we have an isomorphism of Halmos algebras $\gamma_f : R_f \to R_{f^\alpha}$.*

Proof.

Proceed from the corresponding diagrams $*$ and $**$. Given a set X, take a set T consisting of one element $u \in \mathrm{Hal}_{\Phi_1\Theta}(X)$. In this case $T^f = \mathrm{Val}_f(u)$. We have $Ct_f(X,T) = (X, \mathrm{Val}_f(u))$,

$$(X, \mathrm{Val}_f(u))^{\widetilde{\gamma}_f} = Ct_{f^\alpha}(X, u^\beta) = (X, (u^\beta)^{f^\alpha}) = (X, \mathrm{Val}_{f^\alpha}(u^\beta)).$$

Hence, $\widetilde{\gamma}_f$ transfers $\mathrm{Val}_f(u)$ to $\mathrm{Val}_{f^\alpha}(u^\beta)$ for every u, which means that $\widetilde{\gamma}_f$ induces a mapping $\gamma_f : R_f \to R_{f^\alpha}$. It is a homomorphism since Val_f and β are homomorphisms of algebras, and it is an injection since every R_f is a simple algebra [Pl1].

Let now u_1 be an arbitrary element of $\mathrm{Hal}_{\Phi_2\Theta}(X)$. Then the second diagram gives

$$(X, \mathrm{Val}_{f^\alpha}(u_1))^{\widetilde{\gamma}_f^{-1}} = (X, \mathrm{Val}_f(u_1^{\beta'_f})),$$

and

$$(X, \mathrm{Val}_{f^\alpha}(u_1)) = (X, \mathrm{Val}_f(u_1^{\beta'_f}))^{\widetilde{\gamma}_f} = (X, \mathrm{Val}_{f^\alpha}(u))^{\widetilde{\gamma}_f},$$

where $u = u_1^{\beta'_f}$. This implies that $\gamma_f : R_f \to R_{f^\alpha}$ is a surjection. Hence, we have an isomorphism $\gamma_f : R_f \to R_{f^\alpha}$.

7.2 Finite models.

First of all it is clear that for finite models (G, Φ, F) the corresponding KB remains, in general, infinite.

We prove the following main

Theorem 4. *Let the given models be finite. Then the knowledge bases KB_1 and KB_2 are equivalent if and only if there exists a bijection $\alpha : F_1 \to F_2$ such that for every $f \in F_1$ there is an isomorphism $\gamma_f : R_f \to R_{f^\alpha}$.*

Proof.

In one direction the statement is always true. Let now $\gamma_f : R_f \to R_{f^\alpha}$ be an

homomorphisms $\beta_f : \mathrm{Hal}_\Theta(\Phi_1) \to \mathrm{Hal}_\Theta(\Phi_2)$ and $\beta'_f : \mathrm{Hal}_\Theta(\Phi_2) \to \mathrm{Hal}_\Theta(\Phi_1)$ such that the diagrams

$$\begin{tikzcd}
\mathrm{Hal}_\Theta(\Phi_1) \arrow[r, "\beta_f"] \arrow[d, "\mathrm{Val}_f"'] & \mathrm{Hal}_\Theta(\Phi_2) \arrow[d, "\mathrm{Val}_{f^\alpha}"] \\
R_f \arrow[r, "\gamma_f"] & R_{f^\alpha}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Hal}_\Theta(\Phi_1) \arrow[d, "\mathrm{Val}_f"'] & \mathrm{Hal}_\Theta(\Phi_2) \arrow[l, "\beta'_f"'] \arrow[d, "\mathrm{Val}_{f^\alpha}"] \\
R_f & R_{f^\alpha} \arrow[l, "\gamma_f^{-1}"']
\end{tikzcd}$$

are commutative.

Simultaneously, there are functors

$$\widetilde{\beta}_f : L_\Theta(\Phi_1) \to L_\Theta(\Phi_2),$$

$$\widetilde{\beta}'_f : L_\Theta(\Phi_2) \to L_\Theta(\Phi_1).$$

It is left to define the isomorphism of categories $\widetilde{\gamma_f} : K_{\Phi_1\Theta}(f) \to K_{\Phi_2\Theta}(f^\alpha)$ such that the diagrams of the types $*$ and $**$ be commutative.

First we define $\widetilde{\gamma}_f$ on objects and then on morphisms. Take an object (X,T) of the category $L_\Theta(\Phi_1)$ for an arbitrary object (X, A) of the category $K_{\Phi_1\Theta}(f)$ with $T^f = A$. We have $Ct_f(X, T) = (X, T^f) = (X, A)$. Set

$$(X, A)^{\widetilde{\gamma}_f} = (X, T^f)^{\widetilde{\gamma}_f} = (X, \bigcap_{u\in T} \gamma_f \mathrm{Val}_f(u)) =$$

$$(X, \bigcap_{u\in T} \mathrm{Val}_{f^\alpha}(u^{\beta_f})) = (X, T^{\beta_f f^\alpha}).$$

We want to show that this definition does not depend on the choice of the set T with $T^f = A$. Consider first the case when $T_1^f = T_2^f = A$ and the sets T_1 and T_2 are finite. We have: $(X, T_1^f)^{\widetilde{\gamma}_f} = (X, T_1^{\beta_f f^\alpha})$ and $(X, T_2^f)^{\widetilde{\gamma}_f} = (X, T_2^{\beta_f f^\alpha})$.

We need to check that $T_1^{\beta_f f^\alpha} = T_2^{\beta_f f^\alpha}$. Indeed,

$$T_1^{\beta_f f^\alpha} = \bigcap_{u_1\in T_1} \mathrm{Val}_{f^\alpha}(\beta_f u_1)) = \bigcap_{u_1\in T_1} \gamma_f \mathrm{Val}_f(u_1)).$$

Since $\gamma_f : R_f \to R_{f^\alpha}$ is an isomorphism of algebras and T_1, T_2 are finite sets, we can rewrite the expression in the form

$$T_1^{\beta_f f^\alpha} = \gamma_f(\bigcap \mathrm{Val}_f(u_1)) = \gamma_f(\bigcap \mathrm{Val}_f(u_2)) = \bigcap \gamma_f \mathrm{Val}_f(u_2)) = T_2^{\beta_f f^\alpha}.$$

Passing to the general case we proceed from finite models. Every finite model is geometrically noetherian, i.e., if $A = T_1^f = T_2^f$, then in T_1 and T_2 one can find finite subsets T_{01} and T_{02} with ${T_{01}}^f = T_{02}^f = A$. Here, $T_{01}^{\beta_f f_\alpha} = T_{02}^{\beta_f f_\alpha}$. We have to verify that $T_1^{\beta_f f^\alpha} = T_2^{\beta_f f^\alpha}$ and $T_{01}^{\beta_f f_\alpha} = \bigcap_{u_1 \in T_1} \mathrm{Val}_{f^\alpha}(\beta_f u_1)$. We can take a finite subset T_{10} in T_1 such that $T_1^{\beta_f f^\alpha} = T_{10}^{\beta_f f^\alpha}$. Take the union of sets T_{10} and T_{01} and denote it by T_{001}. Then $T_{001}^f = A = T_1^f$, $T_1^{\beta_f f^\alpha} = T_{001}^{\beta_f f^\alpha}$. Analogously, for T_2 take T_{002} and $A = T_{001}^f = T_{002}^f$. Besides that,

$$T_1^{\beta_f f^\alpha} = T_{001}^{\beta_f f^\alpha} = T_2^{\beta_f f^\alpha}.$$

The equality $T_1^{\beta_f f^\alpha} = T_2^{\beta_f f^\alpha}$ gives commutativity of the diagram for objects.

Similarly, we build $\widetilde{\gamma}_f^{-1}$ having γ_f^{-1} and the equality $\widetilde{\gamma}_f^{-1} = \widetilde{\gamma_f^{-1}}$ holds.

Now let us pass to morphisms. Remind first of all that to every homomorphism $s : W(Y) \to W(X)$ there correspond

$$s_*^1 : \mathrm{Hal}_{\Phi_1\Theta}(Y) \to \mathrm{Hal}_{\Phi_1\Theta}(X),$$

$$s_*^2 : \mathrm{Hal}_{\Phi_2\Theta}(Y) \to \mathrm{Hal}_{\Phi_2\Theta}(X).$$

Let the objects (Y, T_2) and (X, T_1) be given in $L_\Theta(\Phi_1)$. Recall that s is admissible for T_2 and T_1 if $s_*^1(u) \in T_1$ for every $u \in T_2$. Here $s_*^1 : (Y, T_2) \to (X, T_1)$ is a morphism. Proceed further from an arbitrary homomorphism $\beta : \mathrm{Hal}_\Theta(\Phi_1) \to \mathrm{Hal}_\Theta(\Phi_2)$. It had been proved that if s is admissible for T_2 and T_1 then the same s is admissible for T_2^β and T_1^β as well, i.e., $s_*^1(u) \in T_1^\beta$ for every $u \in T_2^\beta$. Hence, we have a morphism

$$\widetilde{\beta}(s_*^1) = s_*^2 : (Y, T_2^\beta) \to (X, T_1^\beta).$$

Take now $\beta = \beta_f$ and apply Ct_{f^α}:

$$\mathrm{Ct}_{f^\alpha}(s_*^2) : \mathrm{Ct}_{f^\alpha}(X, T_1^{\beta_X}) \to \mathrm{Ct}_{f^\alpha}(Y, T_2^{\beta_X}).$$

It can be rewritten as

$$\mathrm{Ct}_{f^\alpha}(s_*^2) : (X, T_1^{\beta_X f^\alpha}) \to (X, T_2^{\beta_Y f^\alpha})$$

or

$$\mathrm{Ct}_{f^\alpha}(s_*^2) : (X, T_1^f)^{\gamma_f} \to (X, T_2^f)^{\gamma_f}$$

Let now $T_1^f = A$, $T_2^f = B$. For $s_*^1 : (Y, T_2) \to (X, T_1)$ we have

$$\mathrm{Ct}_f(s_*^1) : (X, T_1^f) \to (Y, T_2^f)$$

and a related morphism

$$\mathrm{Ct}_{f^\alpha}(s_*^2) : (X, T_1^f)^{\gamma_f} \to (Y, T_2^{f_2})^{\gamma_f}.$$

Commutativity of the diagram on morphisms means that

$$\widetilde{\gamma}_f \mathrm{Ct}_f(s_*^1) = \mathrm{Ct}_{f^\alpha}(\widetilde{\beta}_f(s_*^1))$$

for every $s_*^1 : (Y, T_2) \to (X, T_1)$.

Continuing consideration of finite models, proceed from the isomorphism $\gamma_f : R_f \to R_{f^\alpha}$ and the corresponding functor $\widetilde{\gamma}_f : K_{\Phi_1\Theta}(f) \to K_{\Phi_2\Theta}(f^\alpha)$. This functor had been defined on the objects, and now we are going to define it on morphisms.

Let $\tau : (X, A) \to (Y, B)$ be a morphism in $K_{\Phi_1\Theta}(f)$. This τ appears as follows. A morphism

$$s_*^1 : \mathrm{Hal}_{\Phi_1\Theta}(Y) \to \mathrm{Hal}_{\Phi_1\Theta}(X)$$

corresponds to $s : W(Y) \to W(X)$. If now $A = T_1^f$, $B = T_2^f$ and s_*^1 is admissible for T_2 and T_1 then we have $\widetilde{s}_*^1 : (X, A) \to (Y, B)$. We may say that $\tau = \widetilde{s}_*^1$ for some s_*^1.

Define

$$\widetilde{\gamma}_f(\widetilde{s}_*^1) = \widetilde{s}_*^2 : (X, T_1^f)^{\widetilde{\gamma}_f} \to (Y, T_2^f)^{\widetilde{\gamma}_f}.$$

Here,

$$(X, T_1^f)^{\widetilde{\gamma}_f} = (X, T_1^{\beta_f f^\alpha}),$$

$$(Y, T_2^f)^{\widetilde{\gamma}_f} = (Y, T_2^{\beta_f f^\alpha})$$

do not depend on the choice of T_1 and T_2 with $T_1^f = A$ and $T_2^f = B$. Check further that $\widetilde{\gamma}_f : K_{\Phi_1\Theta}(f) \to K_{\Phi_2\Theta}(f^\alpha)$ determined in such a way is in fact a functor and this functor provides commutativity of the diagram on morphisms.

Note first of all that the definition of $\widetilde{\gamma}_f$ on morphisms can be rewritten as

$$\widetilde{\gamma}_f(\mathrm{Ct}_f(s_*^1)) = (\mathrm{Ct}_{f^\alpha}(s_*^2))$$

Take two morphisms $\widetilde{s^1_{1*}} = \mathrm{Ct}_f(s^1_{1*})$ and $\widetilde{s^1_{2*}} = \mathrm{Ct}_f(s^1_{2*})$ and consider the product

$$\widetilde{s^1_{1*}}\widetilde{s^1_{2*}} = \mathrm{Ct}_f(s^1_{1*})\mathrm{Ct}_f(s^1_{2*}) = \mathrm{Ct}_f(s^1_{2*}s^1_{1*}) = \widetilde{s^1_{2*}s^1_{1*}} = (\widetilde{s_2s_1})^1_*.$$

Apply $\widetilde{\gamma}_f$:

$$\widetilde{\gamma}_f((\widetilde{s_2s_1})^1_*) = ((\widetilde{s_2s_1})^2_*) = \widetilde{s^2_{2*}s^2_{1*}} = \widetilde{s^2_{1*}}\widetilde{s^2_{2*}} = \widetilde{\gamma}_f(\widetilde{s^1_{1*}})\widetilde{\gamma}_f(\widetilde{s^1_{2*}}).$$

Now check the commutativity of the diagram

$$\begin{array}{ccc} L_\Theta(\Phi_1) & \xrightarrow{\widetilde{\beta}_X} & L_\Theta(\Phi_2) \\ {\scriptstyle \mathrm{Ct}_f}\downarrow & & \downarrow{\scriptstyle \mathrm{Ct}_{f^\alpha}} \\ K_{\Phi_1\Theta}(f) & \xrightarrow{\widetilde{\gamma}_f} & K_{\Phi_1\Theta}(f^\alpha) \end{array}$$

Take a morphism $s^1_* : (Y, T_2) \to (X, T_1)$ in $L_\Theta(\Phi_1)$. We have

$$\widetilde{\beta_X}(s^1_*) : (Y, T_2^{\beta_X}) \to (X, T_1^{\beta_X}),$$

and

$$\mathrm{Ct}_{f^\alpha}\widetilde{\beta_X}(s^1_*) : (X, T_1^{\beta_X f^\alpha}) \to (Y, T_2^{\beta_X f^\alpha}).$$

Rewrite it as

$$\mathrm{Ct}_{f^\alpha}\widetilde{\beta_X}(s^1_*) : (X, T_1^f)^{\widetilde{\gamma}_f} \to (Y, T_2^f)^{\widetilde{\gamma}_f}.$$

Further,

$$\mathrm{Ct}_f(s^1_*) : (X, T_1^f) \to (Y, T_2^f),$$

$$\widetilde{\gamma}_f\mathrm{Ct}_f(s^1_*) : (X, T_1^f)^{\widetilde{\gamma}_f} \to (Y, T_2^f)^{\widetilde{\gamma}_f}.$$

Check now the equality

$$\widetilde{\gamma}_f\mathrm{Ct}_f(s^1_*) = \mathrm{Ct}_{f^\alpha}\widetilde{\beta}_X(s^1_*)$$

for every s^1_*. We have

$$\widetilde{\gamma}_f\mathrm{Ct}_f(s^1_*) = \widetilde{\gamma}_f(\widetilde{s}^1_*) = \widetilde{s}^2_*,$$

$$\mathrm{Ct}_{f^\alpha}\widetilde{\beta}_X(s^1_*) = \mathrm{Ct}_{f^\alpha}(s^2_*) = \widetilde{s}^2_*.$$

This gives commutativity of the diagram $*$ of morphisms, i.e.,

$$\widetilde{\gamma}_f\mathrm{Ct}_f = \mathrm{Ct}_{f^\alpha}\widetilde{\beta}_X.$$

The same can be done for the functor $\widetilde{\gamma_f^{-1}} = \widetilde{\gamma}_f^{-1}$ and the second commutative diagram $**$ that finishes the proof of the theorem.

7. Additional remarks.

7.1. Let us look at the definition of equivalence from the general perspective of category theory. Given two functors $\varphi_1 : C_1 \to C_1^0$ and $\varphi_2 : C_2 \to C_2^0$, we say that C_1 and C_2 are equivalent in respect to φ_1 and φ_2, if there is an isomorphism $\psi : C_1^0 \to C_2^0$ and functors $\psi_1 : C_1 \to C_2$, $\psi_2 : C_2 \to C_1$ with the commutative diagrams

$$\begin{tikzcd} C_1 \arrow[r, "\psi_1"] \arrow[d, "\varphi_1"'] & C_2 \arrow[d, "\varphi_2"] \\ C_1^0 \arrow[r, "\psi"] & C_2^0 \end{tikzcd}$$

$$\begin{tikzcd} C_1 \arrow[d, "\varphi_1"'] & C_2 \arrow[l, "\psi_2"'] \arrow[d, "\varphi_2"] \\ C_1^0 & C_2^0 \arrow[l, "\psi^{-1}"'] \end{tikzcd}$$

Usual equivalence of categories is equivalence in respect to the transition to skeletons of categories. In our situation we may say that equivalence of knowledge bases means that there exists equivalence of categories of description of knowledge in respect to transition to the categories of knowledge content.

7.2. Let us return to the definition of knowledge bases with multi-models (G_1, Φ_1, F_1) and (G_2, Φ_2, F_2), and let the bijection $\alpha : F_1 \to F_2$ determine equivalence of the corresponding KB_1 and KB_2. Assume that two instances f_1 and f_2 from F_1 are connected by a commutative diagram

$$\begin{tikzcd} \mathrm{Hal}_\Theta(\Phi_1) \arrow[r, "\mathrm{Val}_{f_1}"] \arrow[dr, "\mathrm{Val}_{f_2}"'] & R_{f_1} \arrow[d, "\gamma"] \\ & R_{f_2} \end{tikzcd}$$

where γ is a homomorphism of algebras. We want to evaluate the relation between f_1^α and f_2^α.

Proceed from the diagrams

$$\begin{tikzcd} \mathrm{Hal}_{\Phi_1\Theta} \arrow[r, "\beta_f"] \arrow[d, "\mathrm{Val}_f"'] & \mathrm{Hal}_{\Phi_2\Theta} \arrow[d, "\mathrm{Val}_{f^\alpha}"] \\ R_f \arrow[r, "\gamma_f"] & R_{f^\alpha} \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Hal}_{\Phi_1\Theta} \arrow[d, "\mathrm{Val}_f"'] & \mathrm{Hal}_{\Phi_2\Theta} \arrow[l, "\beta'_f"'] \arrow[d, "\mathrm{Val}_{f^\alpha}"] \\ R \arrow[r, phantom, "\gamma_f^{-1}"] & R \end{tikzcd}$$

$$\begin{tikzcd} R_{f_1} \arrow[r, "\gamma_{f_1}"] \arrow[d, "\gamma"'] & R_{f_1^\alpha} \arrow[d, "\gamma^\alpha"] \\ R_{f_2} \arrow[r, "\gamma_{f_2}"] & R_{f_2^\alpha} \end{tikzcd}$$

Here,

$$\gamma \mathrm{Val}_{f^1} = \mathrm{Val}_{f^2}, \qquad \gamma^\alpha = \gamma_{f_2} \gamma \gamma_{f_1}^{-1}$$

and

$$\gamma^\alpha \mathrm{Val}_{f_1^\alpha} = \gamma^\alpha \gamma_{f_1} \mathrm{Val}_{f_1} \beta'_{f_1} = \gamma_{f_2} \gamma \mathrm{Val}_{f_1} \beta'_{f_1} = \gamma_{f_2} \mathrm{Val}_{f_2} \beta'_{f_1} = \mathrm{Val}_{f_2^\alpha} \beta_{f_2} \beta'_{f_1}.$$

Hence, $\gamma^\alpha \mathrm{Val}_{f^{1\alpha}} = \mathrm{Val}_{f_2^\alpha} \beta_{f_2} \beta'_{f_1}$, i.e., the connection is twisted by the product $\beta_{f_2} \beta'_{f_1}$.

At last, let us note that from the diagrams above follow the natural identities:

1. $\mathrm{Val}_f(u) = \mathrm{Val}_f(\beta'_f \beta_f(u))$ for every $u \in \mathrm{Hal}_\Theta(\Phi_1)$.
2. $\mathrm{Val}_{f^\alpha}(u) = \mathrm{Val}_{f^\alpha}(\beta_f \beta'_f(u)$ for every $u \in \mathrm{Hal}_\Theta(\Phi_2)$.

7.3. Note that the equivalence condition of two knowledge bases in the case of finite multi-models can be formulated in terms of these multi-models (cf. [PTP]).

Definition 2. *Let the models* (G_1, Φ_1, f_1) *and* (G_2, Φ_2, f_2) *be given. Let* $\mathrm{Aut}(f_1)$ *and* $\mathrm{Aut}(f_2)$ *be the corresponding groups of automorphisms. The models* (G_1, Φ_1, f_1) *and* (G_2, Φ_2, f_2) *are called automorphic equivalent if there exists an isomorphism of algebras* $\delta : G_1 \to G_2$ *such that*

$$\mathrm{Aut}(f_2) = \delta \mathrm{Aut}(f_1) \delta^{-1}.$$

Definition 3. *Let the multi-models* (G_1, Φ_1, F_1) *and* (G_2, Φ_2, F_2) *be given. These multi-models are called automorphic equivalent if there exists a bijection* $\alpha : F_1 \to F_2$ *such that for every* $f \in F_1$ *the models* (G_1, Φ_1, f) *and* (G_2, Φ_2, f^α) *are automorphic equivalent.*

It is natural to define an isomorphism of multi-models with the same set of relations Φ_1 and Φ_2. An isomorphism of multi-models implies their automorphic equivalence. Evidently, the inverse statement is not true.

Let the knowledge bases $KB_1 = KB(G_1, \Phi_1, F_1)$ and $KB_2 = KB(G_2, \Phi_2, F_2)$ with the finite multi-models be given.

Theorem 5. *The knowledge bases* $KB_1 = KB(G_1, \Phi_1, F_1)$ *and* $KB_2 = KB(G_2, \Phi_2, F_2)$ *are informationally equivalent if and only if the corresponding models are automorphic equivalent.*

The proof of this theorem is parallel to the proof of the corresponding theorem in [PTP] and uses the Galois-Krasner theory in the given variety of algebras Θ [Pl1]. Theorem 5 provides an algorithm for the informational equivalence verification.

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